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14. ABSTRACT The primary objectives of this research were to develop reduced-order modeling methodologies for control and optimization applications. This was achieved through three specific research goals. First of all, we focused on improving the accuracy of reduced-order model simulations by including closure terms (that can be implemented in practical applications). Secondly, we used sensitivity analysis to improve the accuracy of models over ranges of parameters. Finally, we developed a better methodology for using reduced-order modeling methods for feedback control calculations. We made significant advances in each of these three areas.					
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# **REDUCED-ORDER MODELING FOR OPTIMIZATION AND CONTROL OF COMPLEX FLOWS**

AFOSR FA9550-08-1-0136

## **FINAL REPORT**

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### **OBJECTIVES**

The primary objectives of this research were to develop reduced-order modeling methodologies for control and optimization applications. This was achieved through three specific research goals. First of all, we focused on improving the accuracy of reduced-order model simulations by including closure terms (that can be implemented in practical applications). Secondly, we used sensitivity analysis to improve the accuracy of models over ranges of parameters. Finally, we developed a better methodology for using reduced-order modeling methods for feedback control calculations. We made significant advances in each of these three areas.

### **ACCOMPLISHMENTS**

#### **Overview**

Our research program focused on reduced-order models for large-scale systems with an emphasis on fluids and application to control and optimization problems. In this summary, we will focus on three results that correspond to the three main tasks outlined in our original proposal. In the first section, we provide an overview of our progress introducing closure models for POD / Galerkin dynamical systems based on the advances in the turbulence modeling community. Results are presented for Smagorinsky closure models in two-dimensional and three-dimensional flows about circular cylinders. This includes those corresponding to a multi-level discretization approach to make this closure approach feasible. In the second section, we highlight the use of sensitivity analysis to improve the accuracy of reduced-order models over ranges in parameters. Each of these two approaches (closure and sensitivity) provide significant improvements to reduced-order models making them more attractive as surrogates in optimal design problems. In the third section, we present a novel approach to solving flow control problems by developing reduced-order models specifically for Lyapunov and Chandrasekhar equations (rather than designing controllers for reduced-order models).

## Closure for POD Models

Accurate modeling of complex flows using the proper orthogonal decomposition (POD) requires closure terms that account for the discarded (truncated) basis functions. Prior to this research, closure models were either based on tuning model coefficients to match data or based on simplified turbulence models. The latter is due to the desire to build models that can be evaluated rapidly. However, as our preliminary studies have shown, introducing models based on modern turbulent flow practices leads to more accurate reduced-order models at a significantly higher computational cost. A survey of our closure modeling approaches appear in papers #7, 10, 11, 18, 24, 39, and 47. An enabling method for implementing our closure models is two-level (or multi-level) discretization of the nonlinear closure terms. The general approach is to use coarser meshes to discretize the nonlinear closure terms involving the lower order POD modes. We highlight the main results from this study using a two-dimensional flow past a cylinder at Reynolds number  $Re = 200$  and a three-dimensional flow past a cylinder at Reynolds number  $Re = 1000$  (details can be found in paper # 47).

We approximate the flow velocity  $\mathbf{v}$  using a centering trajectory  $\bar{\mathbf{v}}$  and a linear combination of POD modes  $span\{\phi_j\}_{j=1}^r$ ,

$$\mathbf{v}(x, t) \approx \mathbf{v}_r(x, t) \equiv \bar{\mathbf{v}}(x) + \sum_{j=1}^r \phi_j(x) a_j(t).$$

Substitution of this form into the weak form of the Navier-Stokes equations leads to a low-order dynamical system of the form

$$\dot{a} = b + Aa + a^T Ba, \quad a_j(0) = \langle \phi_j, \mathbf{v}(\cdot, 0) - \bar{\mathbf{v}}(\cdot) \rangle.$$

However, when comparing coefficients  $a_j$  with projected quantities  $\tilde{a}_j \equiv \langle \phi_j, \mathbf{v}(\cdot, t) - \bar{\mathbf{v}}(\cdot) \rangle$ , it is obvious that one needs to model the influence of the discarded POD modes  $\{\phi_{r+1}, \dots\}$  in the dynamical system. This is more obvious as the Reynolds number increases. In the study below, we use a standard Smagorinsky model to account for the dissipative effect of the smaller scale structures in the flow. We are currently preparing a study that compares a variety of closure models using the multi-level discretization approach.

In Figure 1, we present the coefficients  $\tilde{a}_1$  and  $\tilde{a}_3$  as well as the corresponding coefficients using the standard POD / Galerkin approach for the two-dimensional cylinder flow problem (labeled  $a^{POD}$ ). Note there is a poor agreement between these two models (integrated over 1000 shedding cycles).

We use the Smagorinsky closure model account for the discarded modes and present the evolution of the first and third coefficients in Figure 2. These coefficients have much better agreement with  $\tilde{a}$ , however, introduce a substantial computational cost. This cost can be minimized by employing a multi-level discretization approach (using a 4x coarser mesh for the nonlinear Smagorinsky terms). As seen in Figure 2, there is little difference in the solutions using this modified closure term, but the computational cost is reduced by a factor of nearly 20.

This study was repeated for a more challenging flow field corresponding to three dimensional flow past a cylinder at  $Re = 1000$  (the computational mesh is indicated in Figure 3).

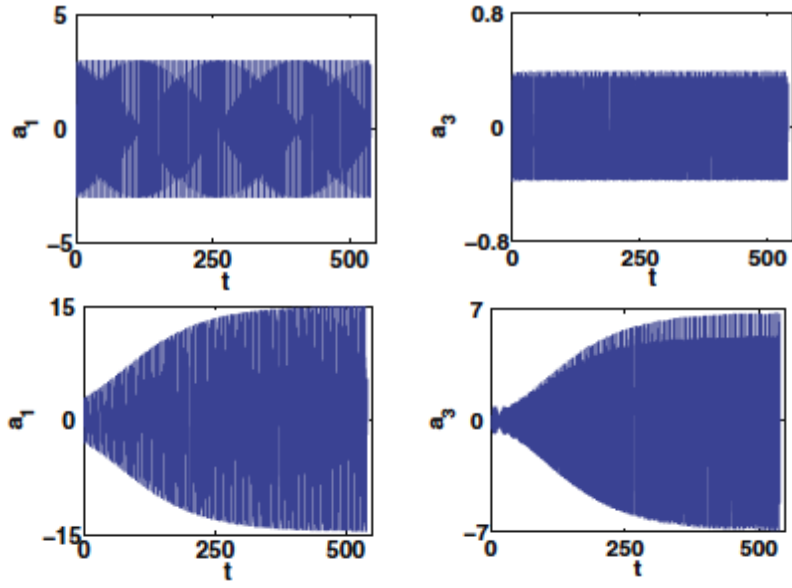


Figure 1: Comparison of  $\tilde{a}$  (top) and  $a^{POD}$  (bottom) coefficients

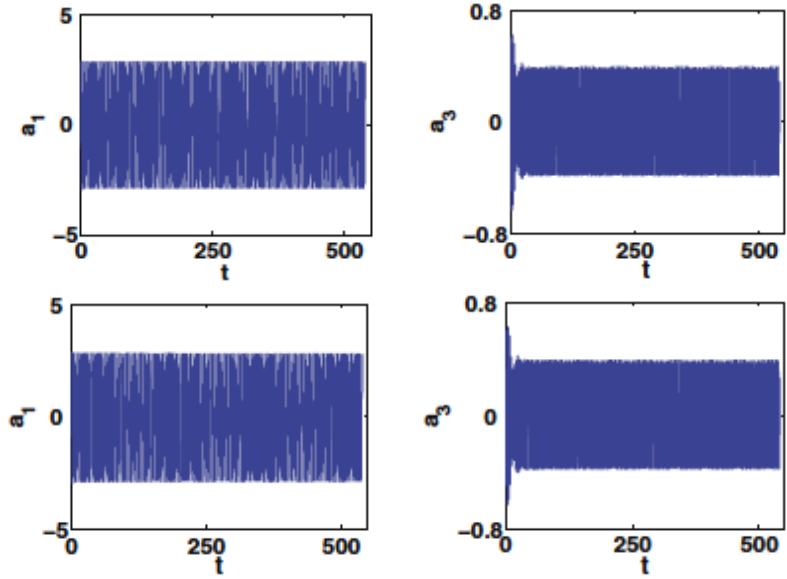


Figure 2: Coefficients of reduced-order model with Smagorinsky closure terms: original mesh (top), coarse mesh (bottom)

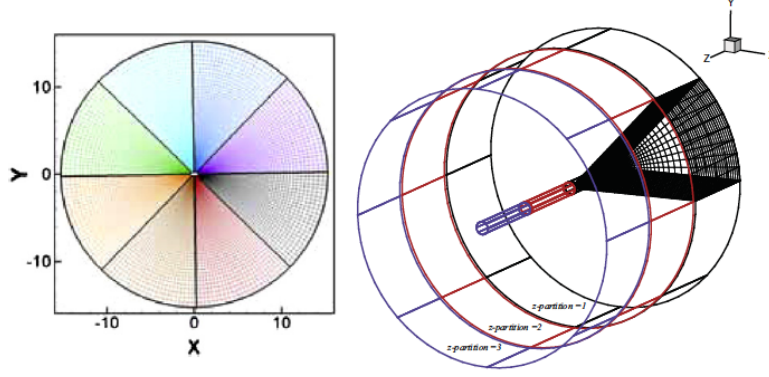


Figure 3: Computational grid for three-dimensional flow past a cylinder

A reduced-order model for  $r = 6$  was computed and found similar improvements with the addition of a Smagorinsky closure model. A table indicating the computational speedup of the two-level methods over the one-level method as well as the relative error in the flow field over one shedding cycle is given below.

coarsening level	speedup factor	error
1	1	$4.46 \times 10^{-2}$
2	5.22	$4.52 \times 10^{-2}$
4	24.18	$4.73 \times 10^{-2}$

### Sensitivity Analysis of POD Modes

This research led to an extensive study on the benefits of using sensitivity analysis to improve the quality of reduced order models from a single simulation. The results of this research is contained in references #5, 9, 25, 30, 31, 32, 33, 34, 35, and 42. We consider a POD model for a flow with a geometric parameter change. The flow again consists of two dimensional flow past a square cylinder, but the angle of the square to the fixed incoming flow and channel walls is parameter dependent. First of all, we introduce a mesh warping function to transform the domain at  $\gamma = 22.5^\circ, \alpha = 0^\circ$  to a range of parameters  $\alpha$  (see Figure 4). This mesh warping function is shown for  $\alpha = -22.5^\circ, 0^\circ$ , and  $22.5^\circ$ , respectively in Figure 5.

This mesh warping function allows us to map POD bases to different geometric configurations as well as define Lagrangian flow sensitivity analysis, to describe how the flow changes with changes to the geometry. This sensitivity analysis can be used to extrapolate the nominal POD basis computed at  $\alpha = 0^\circ$  to a range of parameters. We also consider building reduced-order models that incorporate both the original POD basis as well as the geometric sensitivity of the basis to  $\alpha$  (an example of a POD basis function and the sensitivity of the POD basis function with respect to a parameter are provided in Figure 7).

We compute the relative error (vs. direct numerical simulation) in reduced order models of order 12 over one shedding period and plot the results in Figure 6. We note that the extrapolated basis provides a significant improvement in the reduced order models (nearly an order of magnitude in relative error) at small parameter changes where the extrapolation

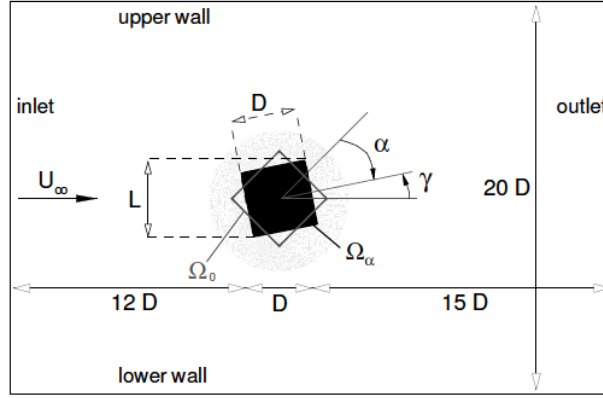


Figure 4: Flow past a cylinder at varying angle of attack,  $\alpha$

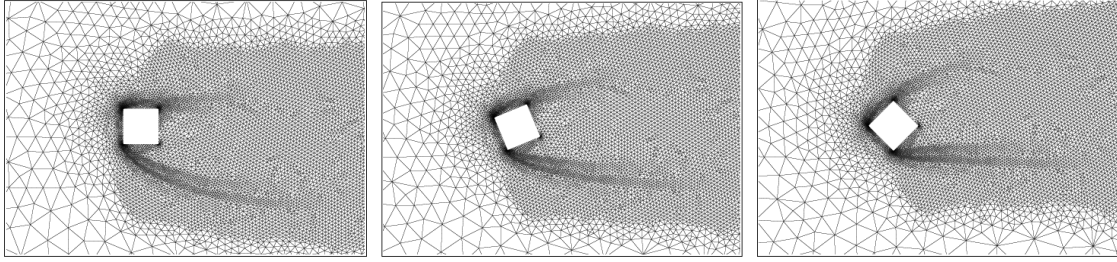


Figure 5: Demonstration of mesh warping function

is expected to be valid. The error in the reduced-order model with the extrapolated basis is comparable to that obtained by projecting the CFD solution onto the basis.

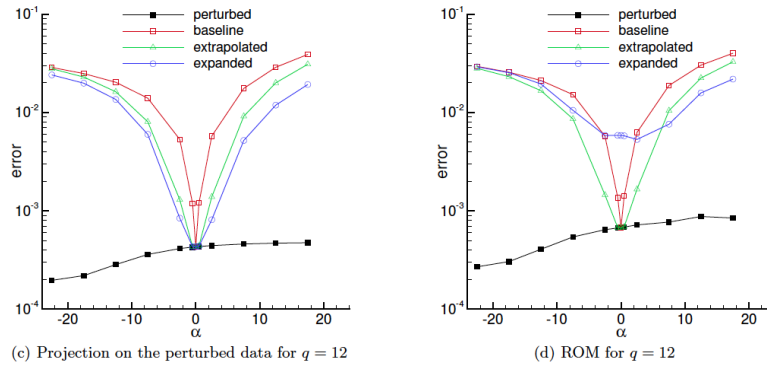
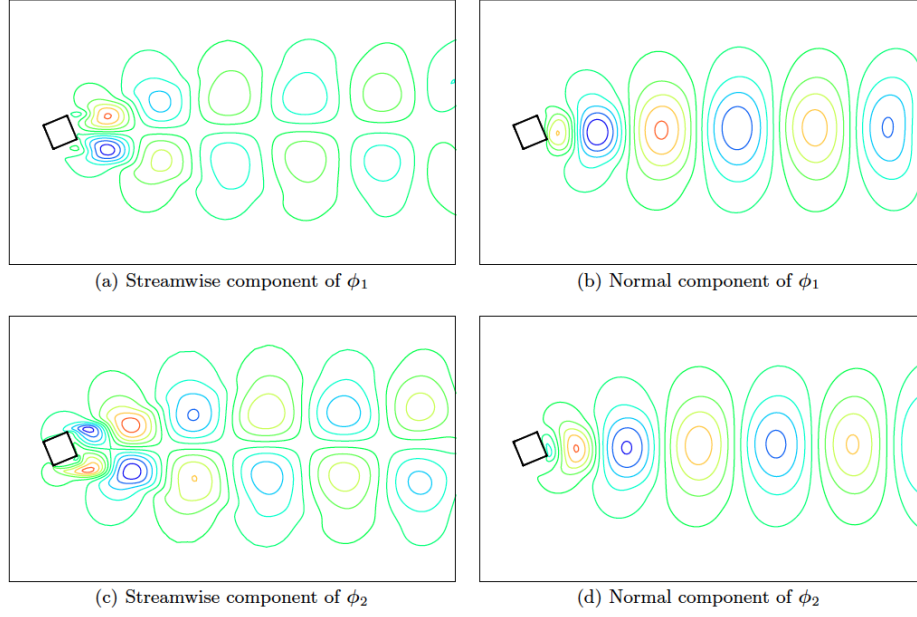
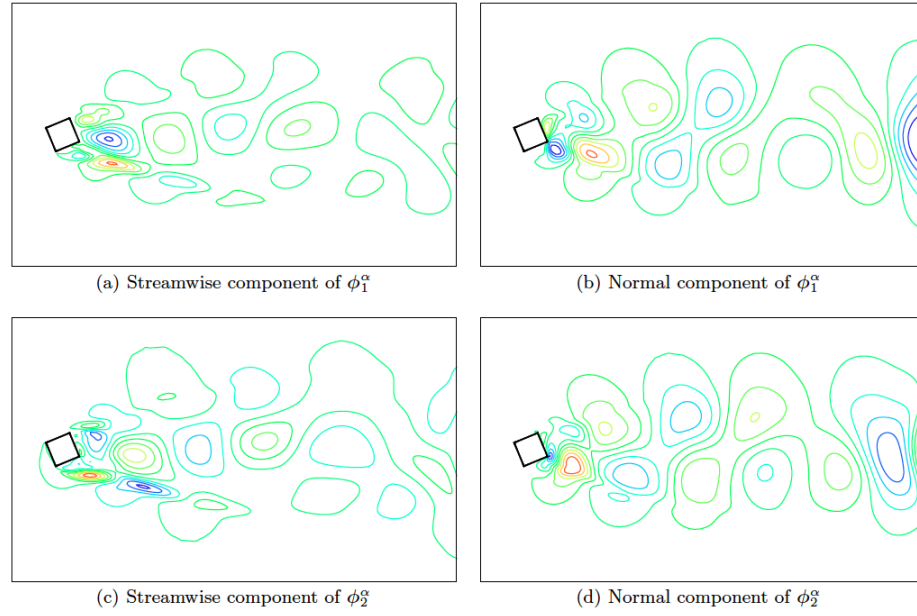


Figure 6: Relative errors in reduced-order models

We note that the example with significant geometric deformation is challenging for most reduced-order modeling capabilities. Combining sensitivity analysis (for varying viscosity and boundary conditions) with POD had more dramatic improvement over larger parametric variation.



(a.) POD modes



(b.) Sensitivity of POD modes with respect to  $\alpha$ .

Figure 7: POD modes and their geometric sensitivity to  $\alpha$

## A Reduced Order Solver for Lyapunov Equations with High Rank Matrices

Consider the Lyapunov equation

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} + \mathbf{Q} = \mathbf{0} \quad (1)$$

where  $\mathbf{A}$  and  $\mathbf{Q}$  are sparse<sup>1</sup>, high rank matrices and  $\mathbf{A}$  is asymptotically stable. We further consider the special case where  $\mathbf{Q}$  is symmetric, leading to symmetric solutions  $\mathbf{P}$ . Furthermore, we are only interested in the action of  $\mathbf{P}$  on one column vector  $\mathbf{b}$  (or a few column vectors in  $\mathbf{B}$ ), in other words, we seek  $\mathbf{P}\mathbf{b}$  (or  $\mathbf{P}\mathbf{B}$ ). Among many control applications, one natural application arises from Kleinman-Newton iterations to solve large scale Riccati equations.

It is well known that the solution  $\mathbf{P}$  to (1) can be written as

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}^T t} \mathbf{Q} e^{\mathbf{A} t} dt.$$

Thus, the action of  $\mathbf{P}$  on a vector  $\mathbf{b}$  can be approximated as the solution to a two point boundary value problem. To see this, we introduce the weakly coupled state/adjoint system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}, & \mathbf{x}(0) &= \mathbf{b} \\ -\dot{\boldsymbol{\lambda}}_\tau &= \mathbf{A}^T \boldsymbol{\lambda}_\tau + \mathbf{Q}\mathbf{x}, & \boldsymbol{\lambda}(\tau) &= \mathbf{0}. \end{aligned} \quad (2)$$

$$(3)$$

Since  $\lim_{\tau \rightarrow \infty} \mathbf{x}(\tau) = \mathbf{0}$ , our approximation is seen in the limiting solution

$$\boldsymbol{\lambda}_\infty(\cdot) = \lim_{\tau \rightarrow \infty} \boldsymbol{\lambda}_\tau(\cdot).$$

Since  $\boldsymbol{\lambda}_\infty(t) = \mathbf{P}\mathbf{x}(t)$ , we arrive at our desired product as  $\boldsymbol{\lambda}_\infty(0) = \mathbf{P}\mathbf{x}(0) = \mathbf{P}\mathbf{b}$ .

A straightforward approach for approximating  $\mathbf{P}\mathbf{b}$  would be to numerically integrate (2) until the solution is essentially zero (guaranteed by the asymptotic stability assumption on  $\mathbf{A}$ ). With the state solution  $\mathbf{x}$  available, the adjoint equation can be integrated backward in time. Since storage would likely be an issue, a checkpointing strategy, eg. [Griewank 2000], could be employed if needed to carry out this approach. However, the cost of integrating equations (2)-(3), even exploiting the fact that only products  $\mathbf{A}\mathbf{x}$ ,  $\mathbf{A}^T \boldsymbol{\lambda}$  and  $\mathbf{Q}\mathbf{x}$  are needed, severely limit the applicability of this approach.

The central strategy proposed in this research is to split the integration above into two pieces. The solution over the interval  $[0, T]$  is integrated accurately until model reduction can be performed to accurately approximate the solution over  $[T, \infty)$ . Then the system over  $[T, \infty)$  reduces to a low order Sylvester equation from which we can obtain an approximate final condition for  $\boldsymbol{\lambda}(T)$ . With the final condition we can integrate the system back to  $t = 0$ . Our reduced order solver is outlined below. Note that for simplicity, we consider approximating the product  $\mathbf{P}\mathbf{b}$  for a column vector  $\mathbf{b}$ . Motivation and parameter selection for the algorithm can be found in papers #8 and 26.

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<sup>1</sup>Alternately, our methodology applied to the case where  $\mathbf{A}$  and  $\mathbf{Q}$  can be decomposed into basic operations involving sparse matrices such that the products  $\mathbf{A}\mathbf{x}$ ,  $\mathbf{A}^T \boldsymbol{\lambda}$  and  $\mathbf{Q}\mathbf{x}$  can be computed cheaply

**Algorithm 1 (Reduced Order Lyapunov Solver)** Given stable  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{b} \in \mathbb{R}^{n \times 1}$ ,  $\mathbf{Q} \in \mathbb{R}^{n \times n}$  ( $\mathbf{Q} = \mathbf{Q}^T$ ) as well as parameters  $T$  and  $r \ll n$ .

1. Integrate the state equation (2) from  $\mathbf{x}(0) = \mathbf{b}$  over the interval  $[0, T]$ .
2. Use the proper orthogonal decomposition, or another reduced order basis method, to generate a low  $r$ -dimensional basis for  $\mathbf{x}(t)$  on the time interval  $[T, \infty)$ . This basis is arranged in columns of the matrix  $\mathbf{V} \in \mathbb{R}^{n \times r}$ .
3. Solve the Sylvester equation

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{V}^T \mathbf{A} \mathbf{V} + \mathbf{Q} \mathbf{V} = \mathbf{0}.$$

*Note that this can be done by performing Schur decomposition to the low order matrix  $\mathbf{V}^T \mathbf{A} \mathbf{V}$  and then solving  $r$  linear systems of order  $n$  (Bartels and Stewart).*

4. Integrate the adjoint equation (3) from  $\boldsymbol{\lambda}(T) = \mathbf{P} \mathbf{V}^T \mathbf{x}(T)$  over  $[0, T]$ .

The result is  $\boldsymbol{\lambda}(0) \approx \mathbf{P} \mathbf{b}$ .

This algorithm was tested on a two-dimensional advection diffusion reaction equation resulting from a linearized 2D Burgers equation [Camphouse 2004]. In the table below, we present relative errors of the solution for varying values of integration time  $T$  and reduced-order model size  $r$ .

Mesh size $N = 1413$ .			
$r$	5	10	20
$T = 2$	1.5882e-05	2.0714e-09	4.6295e-13
$T = 4$	1.2096e-08	4.6095e-13	4.6333e-13
$T = 6$	6.5992e-12	4.6319e-13	4.6317e-13

### An Efficient Long-Time Integrator for Chandrasekhar Equations

The Chandrasekhar equations have been posed as a methodology for solving the infinite horizon control problem when the system involves a distributed parameter system. In this case, the Chandrasekhar equations have the form

$$-\dot{\mathbf{K}}(t) = \mathbf{R}^{-1} \mathbf{B}^T \mathbf{L}(t) \mathbf{L}^T(t), \quad \mathbf{K}(0) = \mathbf{0} \in \mathbb{R}^{m \times n} \quad (4)$$

$$-\dot{\mathbf{L}}(t) = (\mathbf{A} - \mathbf{B} \mathbf{K}(t))^T \mathbf{L}(t), \quad \mathbf{L}(0) = \mathbf{C}^T \in \mathbb{R}^{n \times p}. \quad (5)$$

The solution to the regulator problem  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$  is then given by

$$\mathbf{K} = \lim_{t \rightarrow -\infty} \mathbf{K}(t).$$

This approach replaces the need to find the (dense)  $n \times n$  solution to the Riccati equation by the integration of  $(m+p)n$  equations towards a steady state solution. While the reduced storage costs of the Chandrasekhar equations make some large problems tractable that may

not be otherwise solvable, the slow convergence towards a steady state solution magnify the computational costs associated with the integration.

As with the Lyapunov equation above, we have developed an algorithm that uses reduced-order modeling ideas to dramatically reduce the integration time that is required for (4)-(5). Consider the expression (from the derivation of the Chandrasekhar equations)

$$\Pi = \int_T^0 \mathbf{L}(t) \mathbf{L}^T(t) dt + \underbrace{\int_{-\infty}^T \mathbf{L}(t) \mathbf{L}^T(t) dt}_{\Pi_{\text{res}}} . \quad (6)$$

The first term above is integrated using the Chandrasekhar equation, while we approximate the second integral using a low-dimensional basis for  $\mathbf{L}$  (over  $(-\infty, T)$ ). The expression (6) above shows a clear connection between finding a good basis for  $\mathbf{L}$  and a good basis for  $\Pi_{\text{res}}$ .

**Algorithm 2 (Long-Time Integrator for (4)–(5))** *Given  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ .*

1. *Integrate the Chandrasekhar equations until time  $T < 0$ .*
2. *Build a reduced-basis  $\mathbf{V}$  for  $\mathbf{L}(t)$ ,  $t < T$ .*
3. *Solve a reduced Riccati equation for  $\mathbf{P}$ , set  $\tilde{\mathbf{K}}_{\text{res}} = \mathbf{R}^{-1} \mathbf{B}^T \mathbf{V} \mathbf{P} \mathbf{V}^T$ .*
4. *Use  $\mathbf{K}_{\text{res}} = \mathbf{R}^{-1} \mathbf{B}^T \Pi_{\text{res}}$  to compute  $\mathbf{K} = \mathbf{K}(T) + \mathbf{K}_{\text{res}}$ .*

The accuracy of Algorithm 2 (relative error  $E_{\text{rel}}$  for different integration times  $|T|$  and different sizes of the reduced-order model ( $r$ ) was performed. As we expect,  $E_{\text{rel}}$  is reduced with longer integration time and larger model dimension. However, we observe that longer integration times lead to smaller model dimension requirements to get “maximum” accuracy. This supports our rationale that components of higher frequency modes are ultimately eliminated with integration. It is impressive that the relative error of 2.8% obtained at  $T = -10$  can be reduced 5 orders of magnitude with a 3 dimensional model. Comparisons are provided in references #27 and 46.

## Continuing Research

The co-PIs, Jeff Borggaard and Traian Iliescu along with John Paul Roop at North Carolina A& T are currently extending this research toward the rapid simulation of Boussinesq equations. This includes application of the above techniques with the goal of improving the design / control of thermal fluids in buildings as well as other applications in geophysical fluids.

Natural extensions, such as Hermite interpolation of POD bases in parameter space, incorporating other subgridscale and dynamic LES models, and their combination are currently underway. Continuation of the promising flow control approaches discovered in this research are also planned.

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## Project Summary

### Personnel Supported

#### *Faculty*

Jeff Borggaard and Traian Iliescu (co-PIs)

#### *Post-Docs*

Imran Akhtar

#### *Students*

Vitor Leite Nunes, Miroslav Stoyanov, Hans-Werner van Wyk and Zhu Wang

### Dissemination of Research Results

#### *Publications*

During the project period 1 December 2007–31 November 2010, more than 45 publications were submitted or appeared. The publication topics include model reduction, control, fluid dynamics and sensitivity analysis. Authors supported under this contract are indicated in bold.

1. **Akhtar, I., Borggaard, J.** and Burns, J., (2008), Reduced-Order Models for Optimal Control of Vortex Shedding in *Proceedings of the 4th AIAA Flow Control Conference*, AIAA Paper Number 2008-4083.
2. **Akhtar, I., Borggaard, J.** and Burns, J., (2010), High Performance Computing for Energy Efficient Buildings, in *Proceedings of the International Conference on Power Generation Systems and Renewable Energy Technologies (PGSRET)*, Islamabad, Pakistan.
3. **Akhtar, I., Borggaard, J.** and Burns, J., (2011), On a Control Strategy for Fluid Flows using Model Reduction, in *Proceedings of the 8th International Bhurban Conference on Applied Sciences & Technology (IBCAST)*.

4. **Akhtar, I., Borggaard, J., Burns, J. and Zietsman, L.,** (2008), Model-Based Computation of Functional Gains for Feedback Control of Vortex Shedding in *Proceedings of the 2008 ASME International Mechanical Engineering Congress & Exposition*, Paper Number IMECE2008-68950.
5. **Akhtar, I., Borggaard, J. and Hay, A.,** (2010), Shape Sensitivity Analysis in Flow Models Using a Finite-Difference Approach, *Mathematical Problems in Engineering*, 2010, Article ID 209780, 22 pages.
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15. **Borggaard, J.**, Burns, J., Cliff, E., and Zietsman, L., (2009), A PDE Approach to Optimization and Control of High Performance Buildings, in *Proceedings of the Oberwolfach Workshop on Numerical Techniques for Optimization Problems with PDE Constraints*, M. Heinkenschloss, R. H. W. Hoppe and V. Schulz, Eds., January 2009, 205–208.
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17. **Borggaard, J.**, Burns, J., Surana, A. and Zietsman, L., (2009), Control, Estimation and Optimization of Energy Efficient Buildings, in *Proceedings of the 2009 American Control Conference*, Paper Number WeB05.4.
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21. **Borggaard, J.**, **Iliescu, T.** and Roop, J.-P., (2009), A Bounded Artificial Viscosity Large Eddy Simulation Model, *SIAM Journal on Numerical Analysis*, 47 (1), 622–645.
22. **Borggaard, J.**, **Iliescu, T.** and Roop, J.-P., (2009), An Improved Penalty Method for Power-Law Stokes Problems, *Journal of Computational and Applied Mathematics*, 223 (2), 646–658.
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28. **Borggaard, J.**, **Stoyanov, M.** and Zietsman, L., (2010), Linear Feedback Control of a von Kármán Street by Cylinder Rotation, in *Proceedings of the 2010 American Control Conference*, Paper Number FrB06.3.
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39. **Iliescu, T.** and **Wang, Z.**, (2011), Variational Multiscale Proper Orthogonal Decomposition: Convection-Dominated Convection-Diffusion Equations, submitted.
40. T. Ozgokmen, **Iliescu, T.**, P. Fischer, Srinivasan, A. and Duan, J., (2007), Large Eddy Simulation of Stratified Mixing in Two-Dimensional Dam-Break Problem in a Rectangular Enclosed Domain, *Ocean Modelling*, vol. 16, 106–140.
41. T. Ozgokmen, **Iliescu, T.** and Fischer, P., (2009), Large Eddy Simulation of Stratified Mixing in a Three-Dimensional Lock-Exchange System, *Ocean Modelling*, vol. 26, 134–155.
42. Pelletier, D. and Hay, A. and Etienne, S. and **Borggaard, J.**, (2008), The Sensitivity Equation Method in Fluid Mechanics, *European Journal of Computational Mechanics*, 17 (12), 31-61.
43. Pond, K., (2010), Multidimensional Adaptive Quadrature over Simplices, Ph.D. Dissertation, Virginia Tech.
44. Rui, Z., Huang, J. and **Wang, Z.**, Quadrature Formulas for Hypersingular Integrals and Their Extrapolation Formulas, submitted.
45. San, O., Staples, A.E., **Iliescu, T.** and **Wang, Z.**, (2011), Approximate deconvolution parametrization for two-dimensional turbulent geophysical flows,” submitted.
46. **Stoyanov, M.**, (2009), Reduced Order Methods for Large Scale Riccati Equations, Ph.D. Dissertation, Virginia Tech.
47. **Wang, Z.**, **Akhtar, I.**, **Borggaard, J.** and **Iliescu, T.**, (2011), Two-Level Discretizations of Nonlinear Closure Models for Proper Orthogonal Decomposition, *Journal of Computational Physics*, 230 (1), 126–146.

#### *Presentations at meetings, conferences or seminars*

During this project, we have given more than eighty five presentations at meetings, conferences or seminars. This includes short-courses on optimization and reduced-order modeling. In addition, a postdoctoral researcher and many graduate students attended regional meetings and gave presentations on their research involving reduced-order modeling developments, sensitivity analysis, large eddy simulation, and flow control problems.

#### I. Akhtar

1. 38th AIAA Fluid Dynamics Conference and Exhibit, Seattle, WA, (June 2008).
2. AIAA 4th Flow Control Conference, Seattle, WA, (June 2008).
3. ASME International Mechanical Engineering Congress & Exposition, (November 2008).
4. Fall Fluids Symposium, Blacksburg, VA, (November 2008).

5. APS Division of Fluid Dynamics 61st Annual Meeting, San Antonio, TX, (November 2008).
6. 47th AIAA Aerospace Sciences Meeting and Exhibit, Orlando, FL, (January 2009).
7. 19th AIAA Computational Fluid Dynamics Conference, San Antonio, TX, (June 2009).
8. ASCE-ASME-SES Conference on Mechanics and Materials, Blacksburg, VA, (June 2009).
9. SIAM Conference on Control and its Applications, Denver, CO, (July 2009).
10. Fall Fluids Symposium, Blacksburg, VA, (November 2009).
11. ASME International Mechanical Engineering Congress & Exposition, (November 2009).
12. APS Division of Fluid Dynamics 62nd Annual Meeting, Minneapolis, MA, (November 2009).
13. 48th AIAA Aerospace Sciences Meeting and Exhibit, Orlando, FL, (January 2010).
14. SIAM Conference on Parallel Processing and Scientific Computing, Seattle, WA, (February 2010).
15. University of Dayton, Dayton, OH, (March 2010).
16. Wright State University, Dayton, OH, (March 2010). (also met with AFRL researchers)
17. 8th Annual Meeting of the National Postdoctoral Association, Philadelphia, PA, (March 2010).
18. AIAA 5th Flow Control Conference, Chicago, IL (June 2010).
19. SIAM Annual Meeting, Pittsburgh, PA, (July 2010).
20. Fall Fluids Symposium, Blacksburg, VA, (November 2010).
21. International Conference on Power Generation Systems and Renewable Energy Technologies, Islamabad, Pakistan, (November 2010).

#### J. Borggaard

1. **Short Course:** Large Scale Optimization and Design, DoD High Performance Computing Program Office, University of Tennessee Space Institute, Arnold AFB, TN (February 2008).
2. Inverse Problems: Modeling and Simulation, Fethiye, Turkey, (May 2008).

3. 4th AIAA Flow Control Conference, Seattle, WA, (June 2008).
4. ASME International Mechanical Engineering Congress & Exposition, Boston, MA, (November 2008).
5. 47th IEEE Conference on Decision and Control, Cancun, Mexico, (December 2008).
6. SIAM Conference on Computational Science and Engineering (CSE09), Miami, FL, (March 2009).
7. International Conference on Approximation Methods for Design and Control, Buenos Aires, Argentina, (March 2009).
8. AMS Spring Southeastern Sectional Meeting, Raleigh, NC, (April 2009).
9. SIAM Conference on Mathematical and Computational Issues in the Geosciences, Leipzig, Germany, (June 2009).
10. SIAM Conference on Control and its Applications, Denver, CO, (July 2009).
11. American Control Conference, Baltimore, MD, (July 2010).
12. SIAM Annual Meeting, Pittsburgh, PA, (July 2010).
13. Clemson University, Mathematics Colloquium, Clemson, SC, (April 2008).
14. Florida State University, Department of Scientific Computing Colloquium, Tallahassee, FL, (October 2008).
15. Auburn University, Mathematics and Statistics Colloquium, Auburn, AL, (January 2009).
16. University of Pittsburgh, Mathematics Colloquium, Pittsburgh, PA, (February 2009).
17. Goethe Center for Scientific Computing, Goethe University Frankfurt am Main, Germany, (June 2009).
18. Air Force Institute of Technology, Wright-Patterson Air Force Base, OH, (December 2010).
19. AFOSR Computational Mathematics Program Review, Arlington, VA, (August 2008).
20. 12th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, Victoria, British Columbia, Canada, (September 2008).
21. SIAM Conference on Applications of Dynamical Systems, Snowbird, UT, (May 2009).
22. The 2009 Joint ASCE-ASME-SES Conference on Mechanics and Materials, Blacksburg, VA, (June 2009).

23. AFOSR Computational Mathematics Program Review, Arlington, VA, (July 2009).
24. 29th Annual Southeastern-Atlantic Regional Conference on Differential Equations, Mercer, GA, (October 2009).
25. Emerging Topics in Dynamical Systems and Partial Differential Equations (DSPDEs'10), Barcelona, Spain, (June 2010). (poster)
26. American Physical Society, 63rd Annual Meeting of the Division of Fluid Dynamics, Long Beach, CA, (November 2010).

T. Iliescu

1. AMS Spring Central Meeting, Bloomington, IN (April 2008).
2. Mathematical Theory of Networks and Systems, Blacksburg, VA (July 2008).
3. Navier-Stokes Equations: Classical and Generalized Models, Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy (September 2008).
4. SIAM Conference on Computational Science & Engineering, Miami, Florida (March 2009).
5. AMS Spring Southeastern Section Meeting, Raleigh, NC (April 2009).
6. SIAM Conference on Mathematical and Computational Issues in the Geosciences, Leipzig, Germany (June 2009).
7. The Joint ASCE-ASME-SES Conference on Mechanics and Materials, Blacksburg, VA (June 2009).
8. SIAM Annual Meeting, Pittsburgh, PA (July 2010).
9. Florida State University, Department of Scientific Computing, Seminar, Tallahassee, FL (November 2008).
10. Institute for Scientific Computing and Applied Mathematics, Indiana University, (January 2009).
11. Instituto Superior Tecnico, Department of Mathematics, Lisbon, Portugal (May 2009).
12. Fall Fluids Mechanics Minisymposium at Virginia Tech (November 2008).
13. Fall Fluids Mechanics Minisymposium at Virginia Tech (November 2009).
14. Workshop on Model and Data Hierarchies for Simulating and Understanding Climate, IPAM, Los Angeles, CA, (March 2010).
15. Workshop on Transport and Mixing in Complex and Turbulent Flows, IMA, Minneapolis, MN, (April 2010).

16. AFOSR Computational Mathematics Program Review, Arlington, VA (July 2010).

V. Leite Nunes

1. 2011 SIAM Conference on Mathematical Geosciences, Long Beach, CA, (March 2011).
2. 2011 SIAM Student Conference, Clemson University (February 2011).
3. The 30th Southeastern-Atlantic Regional Conference on Differential Equations, Virginia Tech (October 2010).
4. 34th SIAM Southeastern-Atlantic Section Conference, North Carolina State University, (March 2010).
5. SIAM Student Conference 2010, Virginia Tech (February 2010).
6. The Clemson/Pitt/UTK/VT graduate/post graduate SIAM Student conference, Virginia Tech, Blacksburg, VA (February 2009).

H.-W. van Wyk

1. 35th SIAM Southeastern-Atlantic Section Conference, Charlotte, NC, (March 2011).
2. 2011 SIAM Student Conference, Clemson University (February 2011).
3. SIAM Student Conference 2010, Virginia Tech (February 2010).
4. The Clemson/Pitt/UTK/VT graduate/post graduate SIAM Student conference, Virginia Tech, Blacksburg, VA (February 2009).

Z. Wang

1. 2011 SIAM Computational Science and Engineering, Reno, NV (March 2011).
2. 2011 SIAM Student Conference, Clemson University (February 2011).
3. The 30th Southeastern-Atlantic Regional Conference on Differential Equations, Virginia Tech (October 2010).
4. Student Argonne Summer Symposium, Argonne National Laboratory (August 2010).
5. 2010 SIAM Annual Meeting (AN10), Pittsburgh, PA (July 2010).
6. 34th SIAM Southeastern-Atlantic Section Conference, North Carolina State University, (March 2010).
7. SIAM Student Conference 2010, Virginia Tech (February 2010).
8. The First VT Symposium on Reduced-Order Modeling and System Identification, Virginia Tech (February 2010).

9. Fall Fluid Mechanics Symposium, Virginia Tech (November 2009).
10. SIAM Conference on Computational Science & Engineering (CSE09), Miami, FL (March 2009).
11. The Clemson/Pitt/UTK/VT graduate/post graduate SIAM Student conference, Virginia Tech, Blacksburg, VA (February 2009).
12. Project/NExt/Young Mathematician's Network Poster Session in AMS Joint Mathematics Meeting, Washington, DC (January 2009).

### **Interactions and Transitions**

*Air Force Research Laboratory, Wright-Patterson Air Force Base, OH*

Jeff Borggaard spent the summer of 2007 at AFRL/VACA working with Chris Camphouse and James Myatt on a flow control problem. Efficient POD software using algorithms developed in this research and applicable to practical engineering reduced-order model-based control algorithms was provided to the AFRL researchers. Imran Akhtar visited several AFRL researchers during a visit to Dayton in the spring of 2010. This included several meetings with Phil Beran both in Dayton and when Dr. Beran visited Virginia Tech.

A Ph.D. student at the Interdisciplinary Center for Applied Mathematics, Capt. Kevin Pond, started working at AFIT in the fall of 2010. Jeff Borggaard visited AFIT and AFRL in the fall of 2010, including a meeting with Jack Benek to foster collaborations.

### **Honors/Awards**

Jeff Borggaard was awarded an ASEE Summer Faculty Fellowship and spent May-July 2007 at the AFRL Control Sciences Center of Excellence. This occurred just prior to this grant period.